Time-Optimal Path Following with Bounded Joint Accelerations and Velocities

Tobias Kunz and Mike Stilman

Abstract— **This paper presents a method to generate the time-optimal trajectroy that exactly follows a given differentiable joint-space path within given bounds on joint accelerations and velocities. We also present a path preprocessing method to make nondifferentiable paths differentiable by adding circular blends.**

I. Introduction

To deal with the complexity of planning a robot motion, the problem is often subdivided in two or more subproblems. The first subproblem is that of planning a geometric path through the environment, which does not collide with obstacles.

In additional step this geometric path is turned into a time-parametrized trajectory that follows the previouslt planned path within the capabilities of the robot. Preferably we are looking for a near-optimal trajectory according to some optimality criterion. However, in practice the capabilitites of the robot cannot be expressed exactly. Thus, the capabilities of the robot are normally approximated by using some model of the robot and limiting certain quantities. One option is to model the kinetics of the robot and limit the torques that can be applied by the joints. In this paper we are using limits on joint accelerations and velocities instead.

This paper presents a method to generate the time-optimal trajectroy along a given differentiable path within given bounds on joint accelerations and velocities.

The output of a typical geometric path planner is a path in configuration space consisting of continuous straight line segments between waypoints. Such a path is not differentiable at the waypoints. Thus, we are also describing a preprocessing step to make such a path differentiable by adding circular blends.

II. Related Work

The general approach used here was introduced by [3]. Because they considered a more general case, their derivation is more complicated then ours. Also, they searched for switching points solely numerically.

[1] notes that points, where the limit curve is nondifferentiable, are candidate switching points.

[4] thouroughly analyses all possible cases for switching points and describe methods to explicitly calculate switching point candidates for some cases.

[2] notes that there are points on the limit curve where more than one acceleration is possible. But they do not treat this case correctly.

The authors are with the Center for Robotics and Intelligent Machines (RIM) at the Georgia Institute of Technology, Atlanta 30332, USA. Email: tobias@gatech.edu, mstilman@cc.gatech.edu

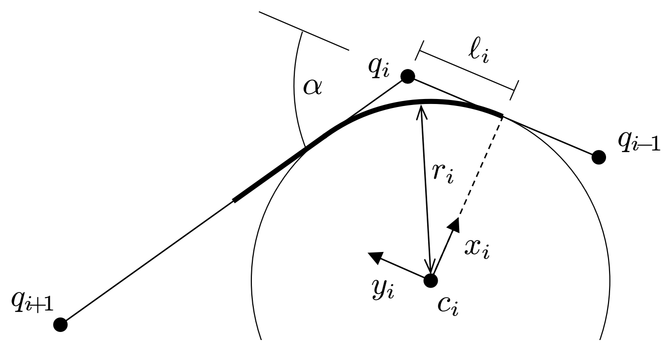

Fig. 1. Circular blend around waypoint

[5] considers joint velocity constraints in addition to torque contraints.

III. Path Preprocessing

Common geometric path planners like PRMs or RRTs usually output the resulting path as a list of waypoints, which are connected by straight lines in configuration space. At a waypoint the path is changing its direction instantaenously and thus is not differentiable. In order to follow such a path exactly, the robot would have to come to a complete stop at every waypoint. This would make the robot motion very slow and look unnatural. Therefore, we adding circular blends around the waypoints, which make the path differentiable. If the path is already differentiable, the preprocessing described in this section can be omitted.

We are looking for a circular segment that starts tangential to the linear path segment before the waypoint and ends tangential to the linear path segment after the waypoint. We also need to assure that the circular segment does not replace more than half of each of the neighboring linear segments. Otherwise we might not get a continuous path. In addition, we allow the enforcement of a maximum deviation from the original path.

First, we are going to define some quantities that are helpful to define the circle. The unit vector y_i pointing from waypoint q_{i-1} to q_i is given by

$$y_i = \frac{q_i - q_{i-1}}{|q_i - q_{i-1}|} \tag{1}$$

The angle α_i between the two adjoining path segments of waypoint q_i is given by

$$\alpha_i = \arccos\left(y_i \cdot y_{i+1}\right) \tag{2}$$

The distance ℓ_i between waypoint q_i and the points where the circle touches the linear segments is given as

$$\ell_i = \min\left\{\frac{|q_i - q_{i-1}|}{2}, \frac{|q_{i+1} - q_i|}{2}, \frac{\delta \sin\frac{\alpha}{2}}{1 - \cos\frac{\alpha}{2}}\right\} \quad (3)$$

where the first two elements give the maximum possible distances such that the circular segment does not replace more than half of the adjoining linear segments and the last element limits the radius to make sure the circular segment stays within a distance δ from waypoint q_i.

Given the quantities above, we can now define the circular segment. The circle is defined by its center c_i, its radius r_i and two vectors x_i and y_i spanning the plane in which the circle lies. The vectors x_i and y_i are orthonormal. x_i points from the center of the circle to the point where the circle toueches the predecessing linear path segment. y_i is the previously defined direction of the predecessing linear segment.

$$r_i = \frac{\ell_i}{\tan\frac{\alpha}{2}} \quad (4)$$

$$c_i = q_i + \frac{y_{i+1} - y_i}{|y_{i+1} - y_i|} \cdot \frac{r_i}{\cos\frac{\alpha}{2}} \quad (5)$$

$$x_i = \frac{q_i - \ell_i y_i - c_i}{|q_i - \ell_i y_i - c_i|} \quad (6)$$

Given these quantities specifying the circle, we can calculate the robot configuration q for any point on the circle, where the point on the circle is given as the arc length s from the start of the circular segment i. Similarly we can calculate the first and second derivative of the function q. Note that these are not time-derivatives but derivatives by s.

$$q(s,i) = c_i + r_i\left(x_i \cos\left(\frac{s}{r_i}\right) + y_i \sin\left(\frac{s}{r_i}\right)\right) \quad (7)$$

$$q'(s,i) = -x_i \sin\left(\frac{s}{r_i}\right) + y_i \cos\left(\frac{s}{r_i}\right) \quad (8)$$

$$q''(s,i) = -\frac{1}{r_i}\left(x_i \sin\left(\frac{s}{r_i}\right) + y_i \cos\left(\frac{s}{r_i}\right)\right) \quad (9)$$

IV. Reduction to 2 Dimensions

The approach we are using was originally proposed in [3], which finds a minimum-time trajectory that satisfies torque limits on the joints. In constrast, we assume acceleration and velocity limits on the joints. Acceleration limits are a special case of torque limits. Using acceleration instead of torque limits results in a simpler problem and a simpler derivation of the following equations. Unlike [3] and like [5], we are also considering velocity limits on the joints.

Because the solution is constrained to follow a given path exactly, the problem can be reduced to 2 dimensions: the position s and velocity $\dot{s} = \frac{ds}{dt}$ along the path. The constraints in the the high-dimensional joint space can be converted into constraints on the scalar path velocity $\dot{s}(s)$ and path acceleration $\ddot{s}(s,\dot{s})$. The constraints on joint velocities result in constraints on the velocity along the path. The constraints on joint accelerations result in constraints on the acceleration and velocity along the path.

A path of length s_f is given as a function $f : [0, s_f] \to \mathbb{R}^n$. The configuration q at some point s along the path is given by

$$q = f(s) \quad 0 \le s \le s_f \quad (10)$$

where s can be an arbitrary parameter. We are going to assume it is the arc length traveled since the start of the path. We can also define the joint velocities and accelerations in respect to the parameter s.

$$\dot{q} = \frac{df}{dt}(s) = \frac{df}{ds}(s)\frac{ds}{dt} = f'(s)\,\dot{s} \quad (11)$$

$$\ddot{q} = f'(s)\,\ddot{s} + f''(s)\,\dot{s}^2 \quad (12)$$

A. Joint Acceleration Limits

We have constraints on the joint accelerations given as

$$-\ddot{q}_i^{\max} \le \ddot{q}_i \le \ddot{q}_i^{\max} \qquad \forall i \in [0, ..., n] \quad (13)$$

where $\ddot{q}_i$ is the ith component of vector $\ddot{q}$. Although the universal quantifier is ommited in the following, all the following inequalities have to hold for all $i \in [0, ..., n]$.

$$-\ddot{q}_i^{\max} \le f_i'(s)\,\ddot{s} + f_i''(s)\,\dot{s}^2 \le \ddot{q}_i^{\max} \quad (14)$$

If $f_i'(s) > 0$:

$$(14) \Leftrightarrow \frac{-\ddot{q}_i^{\max}}{f_i'(s)} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \le \ddot{s} \le \frac{\ddot{q}_i^{\max}}{f_i'(s)} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \quad (15)$$

$$\Leftrightarrow \frac{-\ddot{q}_i^{\max}}{|f_i'(s)|} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \le \ddot{s} \le \frac{\ddot{q}_i^{\max}}{|f_i'(s)|} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \quad (16)$$

If $f_i'(s) < 0$:

$$(14) \Leftrightarrow \frac{-\ddot{q}_i^{\max}}{f_i'(s)} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \ge \ddot{s} \ge \frac{\ddot{q}_i^{\max}}{f_i'(s)} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \quad (17)$$

$$\Leftrightarrow \frac{\ddot{q}_i^{\max}}{|f_i'(s)|} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \ge \ddot{s} \ge \frac{-\ddot{q}_i^{\max}}{|f_i'(s)|} - \frac{f_i''(s)\,\dot{s}^2}{f_i'(s)} \quad (18)$$

If $f_i'(s) = 0$ and $f_i''(s) \ne 0$:

$$(14) \Leftrightarrow \frac{-\ddot{q}_i^{\max}}{|f_i''(s)|} \le \dot{s}^2 \le \frac{\ddot{q}_i^{\max}}{|f_i''(s)|} \quad (19)$$

$$\Leftrightarrow \dot{s} \le \sqrt{\frac{\ddot{q}_i^{\max}}{|f_i''(s)|}} \quad (20)$$

If $f_i'(s) = 0$ and $f_i''(s) = 0$, Eq. 14 is always satisfied.

Eq. 16 and Eq. 18 are equivalent. Thus, the limits on the path acceleration $\ddot{s}$ are

$$\ddot{s}^{\min}(s,\dot{s}) \le \ddot{s} \le \ddot{s}^{\max}(s,\dot{s}) \quad (21)$$

with

$$\ddot{s}^{\min}(s,\dot{s}) = \max_{\substack{i\in[1,\ldots,n] \\ f_i'(s)\ne 0}} \left(\frac{-\ddot{q}_i^{\max}}{|f_i'(s)|} - \frac{f_i''(s)\dot{s}^2}{f_i'(s)}\right) \quad (22)$$

$$\ddot{s}^{\max}(s,\dot{s}) = \min_{\substack{i\in[1,\ldots,n] \\ f_i'(s)\ne 0}} \left(\frac{\ddot{q}_i^{\max}}{|f_i'(s)|} - \frac{f_i''(s)\dot{s}^2}{f_i'(s)}\right) \quad (23)$$

The limit on the path acceleration also constrains the path velocity, because in order for a path velocity to be feasible we need $\ddot{s}^{\min}(s,\dot{s}) < \ddot{s}^{\max}(s,\dot{s})$. This can also be written as

$$\ddot{s}^{\max}(s,\dot{s}) - \ddot{s}^{\min}(s,\dot{s}) \geq 0 \tag{24}$$

$$\Leftrightarrow \left(\frac{\ddot{q}_i^{\max}}{|f_i'(s)|} - \frac{f_i''(s)\dot{s}^2}{f_i'(s)}\right) - \left(\frac{-\ddot{q}_j^{\max}}{|f_j'(s)|} - \frac{f_j''(s)\dot{s}^2}{f_j'(s)}\right) \geq 0$$
$$\forall i,j \in [1,...,n], f_i'(s) \neq 0, f_j'(s) \neq 0 \tag{25}$$

$$\Leftrightarrow -\left(\frac{f_i''(s)}{f_i'(s)} - \frac{f_j''(s)}{f_j'(s)}\right)\dot{s}^2 + \left(\frac{\ddot{q}_i^{\max}}{|f_i'(s)|} + \frac{\ddot{q}_j^{\max}}{|f_j'(s)|}\right) \geq 0$$
$$\forall i,j \in [1,...,n], f_i'(s) \neq 0, f_j'(s) \neq 0 \tag{26}$$

$$\Leftrightarrow -\left|\frac{f_i''(s)}{f_i'(s)} - \frac{f_j''(s)}{f_j'(s)}\right|\dot{s}^2 + \left(\frac{\ddot{q}_i^{\max}}{|f_i'(s)|} + \frac{\ddot{q}_j^{\max}}{|f_j'(s)|}\right) \geq 0$$
$$\forall i \in [1,...,n], j \in [i+1,...,n], f_i'(s) \neq 0, f_j'(s) \neq 0 \tag{27}$$

This gives a set of downward-facing parabolas in $\dot{s}$ horizontally centered around the origin. Each parabola is positive within an interval around 0, which is the interval the feasible velocities may lie in. The positive bound of the interval can be found by setting the parabola equation to zero.

$$-\left|\frac{f_i''(s)}{f_i'(s)} - \frac{f_j''(s)}{f_j'(s)}\right|\dot{s}^2 + \left(\frac{\ddot{q}_i^{\max}}{|f_i'(s)|} + \frac{\ddot{q}_j^{\max}}{|f_j'(s)|}\right) = 0 \tag{28}$$

$$\Leftrightarrow \dot{s} = \sqrt{\frac{\frac{\ddot{q}_i^{\max}}{|f_i'(s)|} + \frac{\ddot{q}_j^{\max}}{|f_j'(s)|}}{\left|\frac{f_i''(s)}{f_i'(s)} - \frac{f_j''(s)}{f_j'(s)}\right|}} \tag{29}$$

The interval of feasible path velocities $\dot{s}$ is defined by the intersection of the feasible intervals of all parabolas. Thus, the upper bound for the path velocity is the minimum of all upper bounds as in Eq. 29. Combining this with the case from Eq. 20, the constraint on the path velocity caused by joint acceleration limits is given by

$$\dot{s} \leq \dot{s}_{\mathrm{acc}}^{\max}(s) \tag{30}$$

with

$$\dot{s}_{\mathrm{acc}}^{\max}(s) = \min\left\{ \min_{\substack{i\in[1,...,n]\\ j\in[i+1,...,n]\\ f_i'(s)\neq 0\\ f_j'(s)\neq 0\\ \frac{f_i''(s)}{f_i'(s)} - \frac{f_j''(s)}{f_j'(s)} \neq 0}} \sqrt{\frac{\frac{\ddot{q}_i^{\max}}{|f_i'(s)|} + \frac{\ddot{q}_j^{\max}}{|f_j'(s)|}}{\left|\frac{f_i''(s)}{f_i'(s)} - \frac{f_j''(s)}{f_j'(s)}\right|}} ,\ \min_{\substack{i\in[1,...,n]\\ f_i'(s)=0\\ f_i''(s)\neq 0}} \sqrt{\frac{\ddot{q}_i^{\max}}{|f_i''(s)|}} \right\} \tag{31}$$

B. Joint Velocity Limits

Constraints on the joint velocities are given as

$$-\dot{q}_i^{\max} \leq \dot{q}_i \leq q_i^{\max} \qquad \forall i \in [1,...,n] \tag{32}$$

Plugging Eq. 11 into Eq. 32 yields

$$-\dot{q}_i^{\max} \leq f_i'(s)\dot{s} \leq q_i^{\max} \qquad \forall i \in [1,...,n] \tag{33}$$

If $f_i'(s) = 0$, Eq. 33 is always satisfied. Otherwise, because $\dot{s} > 0$, it is equivalent to

$$\dot{s} \leq \frac{q_i^{\max}}{|f_i'(s)|} \qquad \forall i \in [1,...,n] \tag{34}$$

Thus, the constraint on the path velocity caused by limits on the joint velocities is given by

$$\dot{s} \leq \dot{s}_{\mathrm{vel}}^{\max}(s) \tag{35}$$

with

$$\dot{s}_{\mathrm{vel}}^{\max}(s) = \min_{\substack{i\in[1,...,n]\\ f_i'(s)\neq 0}} \frac{q_i^{\max}}{|f_i'(s)|} \tag{36}$$

V. Algorithm

1) Start from the start of the path, i. e. $s = 0$ and $\dot{s} = 0$.
2) Integrate forward with maximum acceleration $\ddot{s} = \ddot{s}^{\max}(s,\dot{s})$ until one of the following conditions is met.
   - If $s \geq s_f$, continue from the end of the path, i. e. $s = s_f$ and $\dot{s} = 0$ and go to step 5.
   - If $\dot{s} > \dot{s}_{\mathrm{acc}}^{\max}(s)$, go to step 4.
   - If $\dot{s} > \dot{s}_{\mathrm{vel}}^{\max}(s)$, go to step 3.
3) Follow the limit curve $\dot{s}_{\mathrm{vel}}^{\max}(s)$ until one of the following conditions is met.
   - If $\frac{\ddot{s}^{\max}(s,\dot{s}_{\mathrm{vel}}^{\max}(s))}{\dot{s}} < \frac{d}{ds}\dot{s}_{\mathrm{acc}}^{\max}(s)$, go back to step 2.
   - If $\frac{\ddot{s}^{\min}(s,\dot{s}_{\mathrm{vel}}^{\max}(s))}{\dot{s}} > \frac{d}{ds}\dot{s}_{\mathrm{acc}}^{\max}(s)$, go to step 4.
4) Search along the combined limit curve for the next switching point. See section Section VI for how to find those.
   - Continue from the switching point and go to step 5.
5) Integrate backward with minimum acceleration until the start trajectory is hit. (If the limit curve is hit instead, something went wrong.) The point where the start trajectory is intersected is a switching point. Replace the part of the start trajectory after that switching point with the trajectory just generated.
   - If we transitioned into this step from step 2, halt. The start trajectory reached the end of the path with $s = s_f$ and $\dot{s} = 0$.
   - Otherwise, continue from the end of the start trajectory, which is the switching point found in step 4, and go to step 2.

VI. Switching Points

A. Caused by Accleration Constraints

This section describes how to find switching points along the path velocity limit curve $\dot{s}_{\mathrm{acc}}^{\max}(s)$ caused by constraints on the joint accelerations. We distinguish three cases of these switching points, depending on whether the the curve is continuous and/or differentiable at the switching point.

1) Discontiuous: $\dot{s}_{\text{acc}}^{\max}(s)$ is discontinuous if and only if $f''(s)$ is discontinuous. A discontinuity of $\dot{s}_{\text{acc}}^{\max}(s)$ is a possible switching point if and only if

$$\begin{aligned}(\dot{s}_{\text{acc}}^{\max}(s^-) < \dot{s}_{\text{acc}}^{\max}(s^+) \\ \wedge\ \ddot{s}^{max}(s^-, s_{\text{acc}}^{\max}(s^-)) \geq \frac{d}{ds}\dot{s}_{\text{acc}}^{\max}(s^-)) \\ \vee\ (\dot{s}_{\text{acc}}^{\max}(s^-) > \dot{s}_{\text{acc}}^{\max}(s^+) \\ \wedge\ \ddot{s}^{max}(s^+, s_{\text{acc}}^{\max}(s^+)) \leq \frac{d}{ds}\dot{s}_{\text{acc}}^{\max}(s^+))\end{aligned} \tag{37}$$

2) Continuous and Nondifferentiable: A necessary condition for $\dot{s}_{\text{acc}}^{\max}(s)$ being nondifferentiable is

$$\exists i : f_i'(s) = 0 \tag{38}$$

A sufficient condition for $\dot{s}_{\text{acc}}^{\max}(s)$ being nondifferentiable and s being a possible switching point is

$$\frac{d}{ds}\dot{s}_{\text{acc}}^{\max}(s^-) \leq 0\ \wedge\ \frac{d}{ds}\dot{s}_{\text{acc}}^{\max}(s^+) \geq 0 \tag{39}$$

3) Continuous and Differentiable: Switching points for this case do not exist.

B. Caused by Velocity Constraints

This section describes how to find switching points along the path velocity limit curve $\dot{s}_{\text{vel}}^{\max}(s)$ caused by constraints on the joint velocities. We distinguish two cases of these switching points, depending on whether $\ddot{s}^{\min}(s, \dot{s}_{\text{vel}}^{\max}(s))$ is continuous.

1) Continuous: s is a possible switching point if and only if

$$\ddot{s}^{min}(s, s_{\text{vel}}^{\max}(s)) = \frac{d}{ds}\dot{s}_{\text{acc}}^{\max}(s) \tag{40}$$

2) Discontinuous: $f_i''(s)$ being discontinuos is a necessary condition for $\ddot{s}^{\min}(s, \dot{s}_{\text{vel}}^{\max}(s))$ being discontinuos. s is a possible switching point if and only if

$$(\ddot{s}^{min}(s^-, s_{\text{vel}}^{\max}(s^-)) \geq \frac{d}{ds}\dot{s}_{\text{acc}}^{\max}(s^-)) \tag{41}$$

$$\wedge\ (\ddot{s}^{min}(s^+, s_{\text{vel}}^{\max}(s^+)) \leq \frac{d}{ds}\dot{s}_{\text{acc}}^{\max}(s^+)) \tag{42}$$

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